

# BLOW-UP RESULTS AND SOLITON SOLUTIONS FOR A GENERALIZED VARIABLE COEFFICIENT NONLINEAR SCHRÖDINGER EQUATION

J. ESCORCIA AND E. SUAZO

**ABSTRACT.** In this paper, by means of similarity transformations we study exact analytical solutions for a generalized nonlinear Schrödinger equation with variable coefficients. This equation appears in literature describing the evolution of coherent light in a nonlinear Kerr medium, Bose-Einstein condensates phenomena and high intensity pulse propagation in optical fibers. By restricting the coefficients to satisfy Ermakov-Riccati systems with multiparameter solutions, we present conditions for existence of explicit solutions with singularities and a family of oscillating periodic soliton-type solutions. Also, we show the existence of bright-, dark- and Peregrine-type soliton solutions, and by means of a computer algebra system we exemplify the nontrivial dynamics of the solitary wave center of these solutions produced by our multiparameter approach.

**Keywords.** Soliton-like equations, Nonlinear Schrödinger like equations, Fiber optics, Gross-Pitaevskii equation, Similarity transformations and Riccati-Ermakov systems.

## 1. INTRODUCTION

The study of the nonlinear Schrödinger equation (NLS) with real potential  $V$

$$i\psi_t = -\frac{1}{2}\Delta\psi + V(\mathbf{x},t)\psi + \lambda|\psi|^{2s}\psi, \quad \psi(0, \mathbf{x}) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad \Delta = \sum_{j=1}^n \partial_{x_j x_j} \quad (1.1)$$

has been studied extensively not only for its role in physics, such as in Bose-Einstein condensates and nonlinear optics, but also for its mathematical complexity (for a review of the several results available see [1], [4], [10], [11], [28], [35] and [57]). For the case  $\lambda = -1$ ,  $V \equiv 0$ , and  $ns < 2$  (subcritical case) Weinstein [64] proved that if  $\varphi \in H^1$ , then  $\psi$  exists globally in  $H^1$ . It is also known (see [11], [19] and [57] for a complete review) that NLS for critical ( $ns = 2$ ) and supercritical ( $ns > 2$ ) cases present solutions that become singular in a finite time in  $L^p$  for some finite  $p$ . In [21] singular solutions of the subcritical NLS were presented in  $L^p$ .

In [9] it was proved that if  $\varphi \in \Sigma = \{f \in H^1(\mathbb{R}^n) : \mathbf{x} \rightarrow |\mathbf{x}| f(\mathbf{x}) \in L^2(\mathbb{R}^n)\}$ ,  $V(\mathbf{x},t)$  is real, locally bounded in time and subquadratic in space, and  $\lambda \in \mathbb{R}$ , then the solution of the Cauchy initial value problem exists globally in  $\Sigma$ , provided that  $s < 2/n$  or  $s \geq 2/n$  and  $\lambda \geq 0$ . Also, in [9] it was shown that if  $V(\mathbf{x},t) = b(t)x_j^2$ ,  $b(t) \in C(\mathbb{R};\mathbb{R})$  in (1.1), then there exist blow-up solutions if  $\lambda < 0$  and  $s = 2/n$ . The proof uses the generalized Melher's formula introduced in [13]. In [47] and [58] a generalized pseudoconformal transformation (lens transform in optics [59]) was presented. In this paper, as a first main result we will use a generalized lens transformation to construct solutions with finite-time blow-up in  $L^p$  norm for  $1 \leq p \leq \infty$  of the general variable coefficient nonlinear Schrödinger:

$$i\psi_t = -a(t)\psi_{xx} + (b(t)x^2 - f(t)x + G(t))\psi - ic(t)x\psi_x - id(t)\psi + ig(t)\psi_x + h(t)|\psi|^{2s}\psi. \quad (1.2)$$

*Date:* May 25, 2016.

*1991 Mathematics Subject Classification.* Primary 81Q05, 35C05. Secondary 42A38.

In modern nonlinear sciences some of the most important models are the variable coefficient nonlinear Schrödinger-type ones. Applications include long distance optical communications, optical fibers and plasma physics, see [4], [5], [8], [12], [15], [23], [24], [25], [30], [41], [48], [49], [51], [52], [53], [61], [63], [65] and references therein.

If we make  $a(t) = \Lambda/4\pi n_0$ ,  $\Lambda$  being the wavelength of the optical source generating the beam, and choose  $c(t) = g(t) = 0$ , then (1.2) models a beam propagation inside of a planar graded-index nonlinear waveguide amplifier with quadratic refractive index represented by  $b(t)x^2 - f(t)x + G(t)$ , and  $h(t)$  represents a Kerr-type nonlinearity of the waveguide amplifier, while  $d(t)$  represents the gain coefficient. If  $b(t) > 0$  [48] (resp.  $b(t) < 0$ , see [51]) in the low-intensity limit, the graded-index waveguide acts as a linear defocusing (focusing) lens.

Depending on the selections of the coefficients in equation (1.2), the applications vary in very specific problems (see [61] and references therein):

- Bose-Einstein condensates [28]:  $b(\cdot) \neq 0$ ,  $a$ ,  $h$  constants and other coefficients are zero.
- Dispersion-managed optical fibers and soliton lasers [30], [52] and [53]:  $a(\cdot)$ ,  $h(\cdot)$ ,  $d(\cdot) \neq 0$  are respectively dispersion, nonlinearity and amplification, and the other coefficients are zero.  $a(\cdot)$  and  $h(\cdot)$  can be periodic as well, see [2] and [42].
- Pulse dynamics in the dispersion-managed fibers [41]:  $h(\cdot) \neq 0$ ,  $a$  is a constant and other coefficients are zero.

In this paper to obtain the main results we use a fundamental approach consisting of the use of similarity transformations and the solutions of Riccati Ermakov systems with several parameters inspired by the work in [40]. Similarity transformations have been a very popular strategy in nonlinear optics since the lens transform presented by Talanov [59]; extensions of this approach have been presented in [47] and [58]. Applications include nonlinear optics, Bose-Einstein condensates, integrability of NLS and quantum mechanics, see for example [5], [6], [9], [36] and references therein. E. Marhic in 1978 introduced (probably for the first time) a one-parameter  $\{\alpha(0)\}$  family of solutions for the linear Schrödinger equation of the one-dimensional harmonic oscillator; the use of an explicit formulation (classical Melher's formula [18] and [44]) for the propagator was fundamental. The solutions presented by E. Marhic constituted a generalization of the original Schrödinger wave packet with oscillating width. Also, in [13] a generalized Melher's formula for a general linear Schrödinger equation of the one-dimensional generalized harmonic oscillator of the form (1.2) with  $h(t) = 0$  was presented. For the latter case in [33], [37] and [56], multiparameter solutions in the spirit of Marhic in [40] have been presented. The parameters for the Riccati system arose originally in the process of proving convergence to the initial data for the Cauchy initial value problem (1.2) with  $h(t) = 0$  and in the process of finding a general solution of a Riccati system [54] and [56]. Ermakov systems with solutions containing parameters [33] have been used successfully to construct solutions for the generalized harmonic oscillator with a hidden symmetry [37], and they have also been used to present Galilei transformation, pseudoconformal transformation and others in a unified manner, see [37]. More recently they have been used in [38] to show spiral and breathing solutions and solutions with bending for the paraxial wave equation. In this paper, as a second main result we introduce a family of Schrödinger equations presenting periodic soliton solutions by using multiparameter solutions for Riccati-Ermakov systems. Further, as a third main result we show that these parameters provide a control on the dynamics of solutions for equations of the form (1.2). These results should deserve numerical and experimental studies.

This paper is organized as follows: In Section 2, as an application of a generalized lens transformation and multiparameter solutions for Riccati systems we present conditions to obtain solutions with singularity in finite time in  $L^p$  norm,  $1 \leq p \leq \infty$  for (1.2). Also, we show that through this more general parameter approach we can obtain the same  $L^\infty$  solutions with finite-time blow-up for standard NLS presented in [13] and finite-time blow-up for NLS with quadratic potential. In Section 3, we present a family of soliton solutions for (1.2) presenting bright- and dark-type solitons; this family includes the standard NLS models. This family has multiparameter solutions coming from solutions of a related Ermakov system, extending the results presented in [55], where a Riccati system was used. By the use of these parameters the dynamics of periodic solutions for (1.2) show bending properties, see Figures 1 and 2. In Section 4, again, as an application of generalized lens transformations and an alternative approach to solve the Riccati system (6.1)-(6.6) we present how the parameters provide us with a control on the center axis of the solution of bright and dark soliton solutions for special coefficients in (1.2). Figures 3 and 4 show the bending propagation of the solutions after introducing parameters, extending the results presented in [38] and [55] to (1.2). Also we show that it is possible to construct a transformation that reduces (1.2), with  $a(t) = l_0 = \pm 1$  and  $G(t) = 0$ , to standard NLS with convenient initial data (Lemma 4) in order to assure existence and uniqueness of classical solutions (Proposition 1). As an application we show how the dynamics of the Peregrine soliton solutions of the nonlinear Schrödinger equation consider change when the dissipation,  $d(t)$ , and the nonlinear term,  $h(t)$  change, see Figures 5-8. We have also prepared a Mathematica file as supplemental material where all the solutions for this Section are verified. Finally, in Section 5 we have an appendix recalling the main tools we have used for our results. These tools are a solution with multiparameters of the Riccati system (6.1)-(6.6) and a modification of the transformation introduced in [58]; we have introduced an extra parameter  $l_0 = \pm 1$  in order to use standard solutions for Peregrine-type soliton solutions. Also a 2D version of a generalized lens transformation is recalled. All the formulas from the appendix have been verified previously using computer algebra systems [29].

## 2. FINITE-TIME BLOW-UP FOR NONAUTONOMOUS NONLINEAR SCHRÖDINGER EQUATIONS

In this section as an application of the multiparameter solution for Riccati systems we present conditions needed in order to obtain solutions with singularities in finite time with  $L^\infty$  norm for (1.2). We show that we can obtain the same  $L^\infty$  solutions with finite-time blow-up for standard NLS presented in [13] and finite-time blow-up for the Gross-Pitaevskii equation. Also, as an application of a generalized lens transformation (see section 6.2) in this section we present conditions to obtain solutions with singularity in finite time with  $L^p$  norm for (1.2) for dimensions one and two. We present our first main result:

**Theorem 1** (Solutions with singularity in finite time with  $L^p$  norm,  $1 \leq p \leq \infty$ ). *If the characteristic equation associated to (1.2), i.e*

$$\mu'' - \left( \frac{a'}{a} - 2c + 4d \right) \mu' + 4 \left( ab - cd + d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right) \right) \mu = 0, \quad (2.1)$$

*admits two standard solutions  $\mu_0$  and  $\mu_1$  subject to*

$$\mu_0(0) = 0, \quad \mu_0'(0) = 2a(0) \neq 0 \quad \mu_1(0) \neq 0, \quad \mu_1'(0) = 0, \quad (2.2)$$

*and if we choose  $h(t) = a(t)\beta^2(t)\mu^{2s}(t)$ ,  $\beta$  and  $\mu$  satisfies a solvable Riccati-type system (2.9)-(2.14) (with  $\mu(0), \beta(0) \neq 0$ ), then there exists an interval  $I$  of time such that if  $-\alpha(0) \in \gamma_0(I)$ , then*

(1.2) presents a solution with finite-time blow-up in  $L^p$  norm at  $T^* = \gamma_0^{-1}(-\alpha(0)) \in I$ . Further, solutions present the following explicit form:

(i). If  $p = \infty$ , a solution for (1.2) is given explicitly by

$$\psi_y(x, t) = \frac{1}{\sqrt{\mu(t)}} e^{iS_y(x, t)}, \quad (2.3)$$

where  $y$  is a parameter and  $S_y(t, x) = \alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)$  and  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$ ,  $\delta(t)$ ,  $\varepsilon(t)$  and  $\kappa(t)$  satisfy the Riccati system (2.9)-(2.14).

(ii). If  $1 \leq p < \infty$ , a solution for (1.2) is given explicitly by

$$\psi(x, t) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} \chi(\xi, \tau), \quad \xi = \beta(t)x + \varepsilon(t), \quad \tau = \gamma(t), \quad (2.4)$$

where  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$ ,  $\delta(t)$ ,  $\varepsilon(t)$  and  $\kappa(t)$  are given by (6.10)-(6.22). Additionally,  $\chi$  satisfies

$$i\chi_\tau + \chi_{\xi\xi} + |\chi|^{2s} \chi = 0. \quad (2.5)$$

(iii). (2D case) The natural 2D version of (1.2), the nonlinear equation

$$\begin{aligned} i\psi_t = & -a(\psi_{xx} + \psi_{yy}) + b(x^2 + y^2)\psi - ic(x\psi_x + y\psi_y) - 2id\psi \\ & - (xf_1 + yf_2)\psi + i(g_1\psi_x + g_2\psi_y) - |\psi|^{2s}\psi, \end{aligned} \quad (2.6)$$

where  $a, b, c, d, f_{1,2}$  and  $g_{1,2}$  are real-valued functions of  $t$ , admits an explicit solution with finite-time blow-up of the form

$$\psi = \mu^{-1} e^{i(\alpha(x^2 + y^2) + (\delta_1 x + \delta_2 y) + \kappa_1 + \kappa_2)} \chi(\xi, \eta, \tau), \quad (2.7)$$

where  $\xi = \beta(t)x + \varepsilon_1(t)$ ,  $\eta = \beta(t)y + \varepsilon_2(t)$ ,  $\tau = \gamma(t)$ ,  $h(t) = a(t)\beta^2(t)\mu^{2s}(t)$ , and  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$ ,  $\delta_1(t)$ ,  $\delta_2(t)$ ,  $\kappa_1(t)$ ,  $\kappa_2(t)$ ,  $\varepsilon_1(t)$ ,  $\varepsilon_2(t)$  satisfy the given conditions in Lemma 4. Finally,  $\chi$  is a solution of

$$i\chi_\tau + \chi_{\xi\xi} + \chi_{\eta\eta} + |\chi|^{2s} \chi = 0. \quad (2.8)$$

*Proof.* To prove (i) we follow [13] and look for a solution of the form (2.3). After substituting on (1.2) we obtain the following Riccati system:

$$\frac{d\alpha}{dt} + b(t) + 2c(t)\alpha + 4a(t)\alpha^2 = 0, \quad (2.9)$$

$$\frac{d\beta}{dt} + (c(t) + 4a(t)\alpha(t))\beta = 0, \quad (2.10)$$

$$\frac{d\gamma}{dt} + a(t)\beta^2(t) = 0, \quad (2.11)$$

$$\frac{d\delta}{dt} + (c(t) + 4a(t)\alpha(t))\delta = f(t) + 2\alpha(t)g(t), \quad (2.12)$$

$$\frac{d\varepsilon}{dt} = (g(t) - 2a(t)\delta(t))\beta(t), \quad (2.13)$$

$$\frac{d\kappa}{dt} = g(t)\delta(t) - a(t)\delta^2(t) - \frac{h(t)}{\mu^s(t)}. \quad (2.14)$$

Using (6.10)-(6.22) in the appendix, (2.9)-(2.13) can be solved, but (2.14) absorbs the nonlinearity and must be solved separately. Since there exists an interval  $J$  of time with  $\mu_0(t) \neq 0$  for all  $t \in J$ ,

and  $\mu_0(t)$  and  $\mu_1(t)$  have been chosen to be linearly independent on an interval, let's say  $J'$ , we observe that for  $t \in J \cap J' \equiv I$ , we get

$$\gamma'_0(t) = \frac{W[\mu_0(t), \mu_1(t)]}{2\mu_0^2(t)} \neq 0,$$

and therefore from the general expression for  $\mu$  given by (6.10), see [14], the equation (2.3) will have finite-time blow-up at  $T^* = \gamma_0^{-1}(-\alpha(0)) \in I$ .

Now we proceed to prove (ii). Using the generalized lens transform, see Lemma 2, we can transform the nonautonomous and inhomogeneous nonlinear Schrödinger equation (1.2) into the standard one:

$$i\chi_\tau + \chi_{\xi\xi} + |\chi|^{2s}\chi = 0, \quad (2.15)$$

recalling ([19], [35] and [57]) that the autonomous focusing NLS (2.15) in dimension  $n$  allows solutions of the form  $\chi = e^{i\tau}R(r)$ , where  $r = |\xi|$  and  $R$  is the solution of

$$R''(r) + \frac{n-1}{r}R'(r) - R(r) + R^{2s+1}(r) = 0, \quad R'(0) = 0, \quad R(\infty) = 0. \quad (2.16)$$

In particular, for  $n = 1$ , a solution of (2.15) is given by

$$R(\xi) = (s+1)^{1/2s} \cosh^{-1/s}(s\xi), \quad (2.17)$$

and for all  $t \in R$  and  $p \in [1, \infty]$  we have  $\|R\|_p < \infty$ . Therefore, we obtain a solution with  $L^p$  finite-time blow-up for (1.2) in time of the form

$$\psi(x, t) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} \chi(\xi, \tau) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} e^{i\tau} R(\xi). \quad (2.18)$$

Again, since  $\mu(t) = 2\mu(0)\mu_0(t)(\alpha(0) + \gamma_0(t))$  we can predict a blow-up at  $T^* = \gamma_0^{-1}(-\alpha(0))$ .

To prove (iii), we consider the unique positive radial solution to

$$\Delta Q(\rho) - Q(\rho) + |Q(\rho)|^{1+4/n} Q(\rho) = 0, \quad (2.19)$$

usually referred to as the ground state.  $Q$  vanishes at infinity (see [31] and [60]). Similar to the one dimensional case, Lemma 3 provides a blow-up solution for (1.2) with  $2s = 1 + 4/n$  given by

$$\psi(\rho, t) = \mu^{-1}(t) e^{i(\alpha(t)(x^2+y^2) + (\delta_1(t)x + \delta_2(t)y) + \kappa_1(t) + \kappa_2(t))} Q(\rho).$$

This provides an example of an explicit blow-up solution  $\|\psi(t)\|_p \rightarrow \infty$  as  $t \rightarrow T^* = \gamma_0^{-1}(-\alpha(0))$  for the nonautonomous nonlinear Schrödinger equation (1.2) in the two-dimensional case.  $\square$

The Theorem 1 above allows us to predict in an independent way the  $L^\infty$  solution with finite-time blow-up for standard NLS found in [13] in 2008.

**Example 1.** *If we consider the equation*

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + h|\psi|^{2s}\psi, \quad h = \text{constant}, \quad s \geq 0, \quad (2.20)$$

*we can construct a solution with finite-time blow-up for the corresponding Riccati system (2.9)-(2.14). When we look for a solution of the form (2.3) it is easy to see that  $\alpha_0(t) = \gamma_0(t) = 1/2t$*

and  $\beta_0(t) = -1/t$ , and by (6.10)-(6.16) the solution of the corresponding characteristic equation is given by  $\mu(t) = 2\mu(0)\alpha(0)t + \mu(0)$ . Further we obtain explicitly

$$\alpha(t) = \frac{\alpha(0)}{1 + 2\alpha(0)t}, \quad \beta(t) = \frac{\beta(0)}{1 + 2\alpha(0)t}, \quad \delta(t) = \frac{\delta(0)}{1 + 2\alpha(0)t}, \quad (2.21)$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0)t}{2(1 + 2\alpha(0)t)}, \quad \varepsilon(t) = \varepsilon(0) - \frac{\beta(0)\delta(0)t}{1 + 2\alpha(0)t}. \quad (2.22)$$

The equation (2.14) must be solved separately, and  $\kappa(t)$  is given by

$$\kappa(t) = \kappa(0) - \frac{\delta^2(0)t}{2(1 + 2\alpha(0)t)} - \frac{h}{\alpha(0)}\xi_s(t)$$

with

$$\xi_s(t) = \begin{cases} \frac{1}{(1-s)} \left( \left( \frac{1}{2} + t\alpha(0) \right)^{1-s} - \left( \frac{1}{2} \right)^{1-s} \right), & \text{when } s \neq 1, \\ \ln(1 + 2t\alpha(0)), & \text{when } s = 1. \end{cases} \quad (2.23)$$

Now, choosing  $\alpha(0) = 0$ ,  $|\psi(t, x)| = 1/\sqrt{2}$  is bounded at all times. However, when  $\alpha(0) \neq 0$ , one obtains

$$|\psi(x, t)| = \frac{1}{\sqrt{\frac{1}{2} + t\alpha(0)}}, \quad t \geq 0, \quad (2.24)$$

which is bounded if  $\alpha(0) > 0$ , and blows up at a finite time  $T^* = -1/2\alpha(0)$  if  $\alpha(0) < 0$ . As expected, this result agrees with the prediction of the theorem above, since  $\gamma_0^{-1}(-\alpha(0)) = -1/2\alpha(0)$ .

The following example shows blow-up for the Gross-Pitaevskii equation:

**Example 2.** Let's consider the Gross-Pitaevskii equation

$$i\psi_t = -(\psi_{xx} + \psi_{yy}) + \sum_{j=1}^2 \frac{\Omega(t)}{2} x_j^2 \psi + \lambda |\psi|^2 \psi. \quad (2.25)$$

The characteristic equation associated to Gross-Pitaevskii equation is given by

$$\mu'' + \Omega(t)\mu = 0. \quad (2.26)$$

Assuming  $\Omega(t)$  is such that (2.26) allows two independent solutions  $\mu_0(t)$  and  $\mu_1(t)$  satisfying (2.2), then

$$\alpha_0(t) = \frac{\mu_0'(t)}{2\mu_0(t)}, \quad \beta_0(t) = -\frac{1}{\mu_0(t)}, \quad \gamma_0(t) = \frac{\mu_1(t)}{2\mu_1(0)\mu_0(t)} \quad (2.27)$$

and

$$\delta_0(t) = \varepsilon_0(t) = \kappa_0(t) = 0. \quad (2.28)$$

By Theorem 1, and using (6.10)-(6.22) from the appendix, a solution for (2.15) is given by

$$\psi(\rho, t) = \mu^{-1}(t) e^{i(\alpha(t)(x^2+y^2) + (\delta_1(t)x + \delta_2(t)y) + \kappa_1(t) + \kappa_2(t))} Q(\rho).$$

with  $\mu(t) = 2\mu(0)\mu_0(t)(\alpha(0) + \gamma_0(t))$

Considering  $\alpha(0) = 0$ ,  $\beta(0) = 1$ ,  $\mu_1(0) = 1$  and  $\gamma(0) = 0$ , then

$$\tau = \gamma(t) = \frac{\mu'_0(t)}{2\mu_1(t)}, \quad \mu(t) = \mu(0)\mu_1(t), \quad \beta(t) = \frac{-\beta(0)}{\mu_1(t)}, \quad \alpha(t) = \frac{\mu'_1(t)}{2\mu_1(t)}, \quad (2.29)$$

and considering  $\kappa_i(0) = 0$  and for  $i = 1$  and  $2$  we obtain

$$\delta_i(t) = \frac{\delta_i(0)}{\mu_1(t)}, \quad \varepsilon_i(t) = -\frac{\beta(0)\delta_i(0)\mu_0(t)}{\mu_1(t)}, \quad \kappa_i(t) = -\frac{(\delta_i(0))^2\mu_0(t)}{2\mu_1(t)}.$$

**Remark 1.** Alternatively, we can use in this example the soliton solution  $u(x, t) = e^{it}Q(x)$  to (2.15) so that after applying the pseudoconformal transform we can obtain solutions which blow up in finite time (see a nice discussion in [60]). Therefore, the following is a solution with blow up for (2.15) given by

$$\chi(x, t) = \frac{1}{t^{d/2}}Q\left(\frac{x}{t}\right)e^{i\left(\frac{|x|^2}{2t}\right) - \frac{i}{t}} \quad (2.30)$$

where  $Q$  is a ground state solution of (2.19). Further, the following is a solution for (2.25) given by

$$\psi(x, t) = \frac{e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))}}{\mu(t)\tau}Q\left(\frac{\xi}{\tau}\right)e^{i\frac{|\xi|^2}{2\tau} - \frac{i}{\tau}}, \quad (2.31)$$

where  $\xi = (\beta(t)\mathbf{x} + \varepsilon(t))/\tau$ ,  $\mathbf{x} = (x, y)$  and  $\varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t))$ , making  $\delta_i(0) = 0$ , and after simplification the module of (2.31) is

$$|\psi(x, t)| = \left| \frac{1}{\mu(0)\mu_0(t)}Q\left(\frac{-\beta(0)\mathbf{x}}{\mu_0(t)}\right) \right| \quad (2.32)$$

and then

$$\lim_{t \rightarrow 0} \|\psi(t)\|_p \rightarrow \infty. \quad (2.33)$$

It is possible to predict finite-time blow-up for toy examples:

**Example 3.** If we consider

$$i\psi_t = -\psi_{xx} + \frac{x^2}{4}[\sin^2 t - \cos t]\psi + ix \sin t \psi_x - i \sin t \psi - 3e^{(3-3\cos t)}|\psi|^2\psi, \quad (2.34)$$

the corresponding characteristic equation is given by  $\mu'' - 6 \sin t \mu' + (9 \sin^2 t - 3 \cos t)\mu = 0$ . The two fundamental solutions are given by  $\mu_0(t) = te^{3(1-\cos t)}$  and  $\mu_1(t) = e^{3(1-\cos t)}$ , and also by  $\gamma_0(t) = 1/2t$  and  $\mu(t) = \mu(0)e^{3(1-\cos t)}[2\alpha(0)t + 1]$ . The explicit solution of the form (2.3) will satisfy

$$|\psi(x, t)| = \frac{1}{e^{3(1-\cos t)}\mu(0)(2\alpha(0)t + 1)}, \quad (2.35)$$

showing finite-time blow-up at  $T^* = -1/2\alpha(0)$ . Again, this result agrees with the prediction of the theorem above since  $\gamma_0^{-1}(-\alpha(0)) = -1/2\alpha(0)$ .

### 3. SOLITON SOLUTIONS FOR A GENERALIZED VARIABLE-COEFFICIENT NLS USING ERMAKOV'S SYSTEM

In this section we present a family of Schrödinger-type equations admitting soliton solutions for (1.2). Using a multiparameter solution for the Ermakov system, see section 2, we present bright and dark-type solitons for (1.2), extending the results presented in [55] where a Riccati system was used. We will use Lemma 3. Further, by the use of these multiparameters, the solutions can be periodic with bending propagation as in [38] for the paraxial wave equation. We proceed to prove our second main result.

**Theorem 2** (Construction of solitons using Ermakov's system). *The nonlinear Schrödinger equation with variable coefficients of the form*

$$\begin{aligned} i\psi_t &= -a(t)\psi_{xx} + B(t)x^2\psi - ic(t)x\psi_x - id(t)\psi \\ &\quad - M(t)x\psi + ig(t)\psi_x + L(t)\psi + h(t)|\psi|^2\psi \end{aligned} \quad (3.1)$$

has a soliton-type solution of the form

$$\psi_y(t, x) = \frac{F(\beta(t)x + 2\gamma(t)y + \varepsilon(t))}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t) + \xi(t))}, \quad (3.2)$$

( $y$  is a parameter) where  $F$  satisfies

$$F'' = -\xi_0 F + h_0 F^3, \quad (3.3)$$

and the following balance between coefficients (using (6.32)-(6.38)) has been imposed:

$$B(t) = b(t) - c_0 a(t) \beta^4(t), \quad (3.4)$$

$$M(t) = f(t) + 2c_0 a(t) \beta^3(t) \varepsilon(t), \quad (3.5)$$

$$L(t) = c_0 a(t) \beta^2(t) \varepsilon^2(t), \quad (3.6)$$

$$h(t) = h_0 a(t) \beta^2(t) \mu(t), \quad (3.7)$$

$$\xi(t) = \xi_0(\gamma(t) - \gamma(0)). \quad (3.8)$$

*Proof.* We look for a solution of the form

$$\psi_y = A_y(x, t) e^{iS_y(x, t)} \quad (3.9)$$

with  $S_y(t, x) = \alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t) + \xi(t)$ , and  $y$  is a parameter (we omit the subindex  $y$  in calculations).

Replacing (3.9) in (3.1) and assuming  $A \geq 0$ , we obtain

$$\begin{aligned} iA_t - AS_t &= -a(t)A_{xx} - 2ia(t)A_x S_x + a(t)S_x^2 - ia(t)AS_{xx} \\ &\quad + b(t)x^2A - ic(t)xA_x + c(t)xAS_x - id(t)A \\ &\quad - f(t)xA + ig(t)A_x - g(t)AS_x + h(t)A^3. \end{aligned} \quad (3.10)$$

Taking the complex part, we obtain

$$A_t = -((4a\alpha + c)x + 2a\beta y + 2a\delta - g)A_x - (2a\alpha + d)A, \quad (3.11)$$

taking the real part and equating coefficients as in [13]. We thus obtain the nonlinear ODE

$$aA_{xx} = \frac{d\xi}{dt}A + h(t)A^3. \quad (3.12)$$



Now, using (3.4)-(3.6), we will obtain the Ermakov-type system from (3.10)

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = c_0a\beta^4, \quad (3.13)$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0, \quad (3.14)$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0, \quad (3.15)$$

$$\frac{d\delta}{dt} + (c + 4a\alpha)\delta = f + 2cg + 2c_0a\beta^3\varepsilon, \quad (3.16)$$

$$\frac{d\varepsilon}{dt} = (g - 2a\delta)\beta, \quad (3.17)$$

$$\frac{d\kappa}{dt} = g\delta - a\delta^2 + c_0a\beta^2\varepsilon^2 \quad (3.18)$$

$$\alpha(t) = -\frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}. \quad (3.19)$$

The solution for this system is given by Lemma 3 in the appendix. Therefore, we have obtained an explicit expression for  $S_y(x, t)$  in (3.9).

We proceed to find an expression for  $A(x, t)$ . Using (3.17), we can transform (3.11) into

$$A_t + \left( -\frac{\dot{\beta}}{\beta}x + 2a\beta y - \frac{\dot{\varepsilon}}{\beta} \right) A_x + \frac{\dot{\mu}}{2\mu}A = 0 \quad (3.20)$$

and look for a solution for (3.12) of the form

$$A(x, t) = \frac{1}{\sqrt{\mu}}F(z), \quad z = C_0(t)x + C_1(t)y + C_2(t). \quad (3.21)$$

Anzats (3.21) and Ermakov's system guide us to choose  $C_0(t) = \beta(t)$ ,  $C_1(t) = \gamma(t)$  and  $C_2(t) = \varepsilon(t)$ , and therefore, (3.12) becomes

$$F_{zz} = \frac{\dot{\xi}}{\beta^2(t)a(t)}F + \frac{h(t)}{\mu(t)\beta^2(t)a(t)}F^3. \quad (3.22)$$

From here we obtain conditions (3.7)-(3.8), and  $A$  would be given explicitly by

$$A(x, t) = \frac{F(\beta(t)x + 2\gamma(t)y + \varepsilon(t))}{\sqrt{\mu(t)}}.$$

□

**Remark 2.** Equation (3.3) presents the following classical nonlinear wave configurations, see [55] and references therein.

If  $h_0 < 0$

$$F(z) = \left( \frac{-\xi_0 + \sqrt{\xi_0^2 - 2C_0h_0}}{-h_0} \right)^{1/2} \quad (3.23)$$

$$\times cn \left( (\xi_0^2 - 2C_0h_0)^{1/4} z, \left( \frac{-\xi_0 + \sqrt{\xi_0^2 - 2C_0h_0}}{2\sqrt{\xi_0^2 - 2C_0h_0}} \right)^{1/2} \right),$$

then  $cn(u, k)$  is a Jacobi elliptic function. A familiar special case is obtained with  $C_0 = 0$ , the bright soliton:

$$F(z) = \sqrt{\frac{-2\xi_0}{-h_0}} \frac{1}{\cosh(\sqrt{-\xi_0}z)} \quad (3.24)$$

when  $cn(u, 1) = 1/\cosh u$ .

If  $\xi_0 > 0$

$$F(z) = \left( \frac{\xi_0 + \sqrt{\xi_0^2 - 2C_0h_0}}{h_0} \right)^{1/2} \times sn \left( \left( \frac{C_0h_0}{\xi_0 + \sqrt{\xi_0^2 - 2C_0h_0}} \right) z, \left( \frac{-\xi_0 - \sqrt{\xi_0^2 - 2C_0h_0}}{-\xi_0 + \sqrt{\xi_0^2 - 2C_0h_0}} \right)^{1/2} \right), \quad (3.25)$$

then  $sn(u, k)$  is a Jacobi elliptic function. Another familiar case is obtained with  $C_0 = \xi_0^2/(2h_0)$ , the dark soliton:

$$F(z) = \sqrt{\frac{\xi_0}{h_0}} \tanh \left( \sqrt{\frac{\xi_0}{2}} z \right) \quad (3.26)$$

$sn(u, 1) = \tanh u$ .

**3.1. Family of solutions.** The following family of equations ( $h(0)$ ,  $\mu(0)$ ,  $\alpha(0)$ ,  $\beta(0)$ ,  $\gamma(0)$ ,  $\delta(0)$ ,  $\varepsilon(0)$  and  $\kappa(0)$  are parameters)

$$i\psi_t = -\frac{1}{2}\psi_{xx} + (1-\beta^4)x^2\psi - 2\beta^3(t)\varepsilon(t)x\psi + \beta^2(t)\varepsilon^2(t)\psi + \frac{h(0)\beta^2(0)}{\sqrt{\beta^4(0)\sin^2 t + (2\alpha(0)\sin t + \cos t)^2}} |\psi|^2 \psi, \quad (3.27)$$

with

$$\beta(t) = \frac{\beta(0)}{\sqrt{\beta^4(0)\sin^2 t + (2\alpha(0)\sin t + \cos t)^2}}, \quad (3.28)$$

$$\varepsilon(t) = \frac{\varepsilon(0)(2\alpha(0)\sin t + \cos t) - \beta(0)\delta(0)\sin t}{\sqrt{\beta^4(0)\sin^2 t + (2\alpha(0)\sin t + \cos t)^2}}, \quad (3.29)$$

admits a family of soliton solutions given by

$$\psi_y(x, t) = \frac{F(\beta(t)x + 2\gamma(t)y + \varepsilon(t))}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t) + \xi(t))} \quad (3.30)$$

where  $F$  satisfies (3.3) and

$$\mu(t) = \mu(0)\sqrt{\beta^4(0)\sin^2 t + (2\alpha(0)\sin t + \cos t)^2} \quad (3.31)$$

$$\alpha(t) = \frac{\alpha(0) \cos 2t + \sin 2t (\beta^4(0) + 4\alpha^2(0) - 1)/4}{\beta^4(0) \sin^2 t + (2\alpha(0) \sin t + \cos t)^2}, \quad (3.32)$$

$$\gamma(t) = \gamma(0) - \frac{1}{2} \arctan \frac{\beta^2(0) \sin t}{2\alpha(0) \sin t + \cos t}, \quad (3.33)$$

$$\delta(t) = \frac{\delta(0) (2\alpha(0) \sin t + \cos t) + \varepsilon(0) \beta^3(0) \sin t}{\beta^4(0) \sin^2 t + (2\alpha(0) \sin t + \cos t)^2}, \quad (3.34)$$

$$\begin{aligned} \kappa(t) = & \kappa(0) + \sin^2 t \frac{\varepsilon(0) \beta^2(0) (\alpha(0) \varepsilon(0) - \beta(0) \delta(0)) - \alpha(0) \delta^2(0)}{\beta^4(0) \sin^2 t + (2\alpha(0) \sin t + \cos t)^2} \\ & + \frac{1}{4} \sin 2t \frac{\varepsilon^2(0) \beta^2(0) - \delta^2(0)}{\beta^4(0) \sin^2 t + (2\alpha(0) \sin t + \cos t)^2}, \end{aligned} \quad (3.35)$$

$$\xi(t) = \xi_0(\gamma(t) - \gamma(0)). \quad (3.36)$$

Now we show that this family contains classic soliton examples.

**Example 4.** (*Classical bright soliton*) The equation (3.27) with  $\delta(0) = 0$ ,  $h(0) = -2$ ,  $\beta(0) = 1$ ,  $\mu(0) = 1$  and  $\alpha(0) = \gamma(0) = \varepsilon(0) = \kappa(0) = 0$  becomes the classical NLS

$$i\psi_t = -\frac{1}{2}\psi_{xx} - 2|\psi|^2\psi, \quad (3.37)$$

which admits the bright soliton of the form ( $y$  is velocity)

$$|\psi_y(x, t)|^2 = \text{sech}^2(x - ty). \quad (3.38)$$

**Example 5.** (*Classical dark soliton*) The equation (3.27) with  $\delta(0) = 0$ ,  $h(0) = 2$ ,  $\beta(0) = 1$ ,  $\mu(0) = 1$  and  $\alpha(0) = \gamma(0) = \varepsilon(0) = \kappa(0) = 0$  becomes the classical NLS

$$i\psi_t = -\frac{1}{2}\psi_{xx} + 2|\psi|^2\psi, \quad (3.39)$$

which admits the dark soliton of the form ( $y$  is velocity)

$$|\psi_y(x, t)|^2 = \tanh^2(x - ty). \quad (3.40)$$

The following examples show periodic solutions with bending propagation.

**Example 6.** (*Bright-type soliton*) The equation (3.27) with  $\delta(0)$  a parameter,  $h(0) = -2$ ,  $\beta(0) = 2/3$ ,  $\mu(0) = 1$  and  $\alpha(0) = \gamma(0) = \varepsilon(0) = \kappa(0) = 0$  becomes

$$i\psi_t = -\frac{1}{2}\psi_{xx} + (1 - \beta^4)x^2\psi - 2\beta^3(t)\varepsilon(t)x\psi + \beta^2(t)\varepsilon^2(t)\psi - \frac{8}{9\sqrt{\frac{16}{81}\sin^2 t + \cos^2 t}}|\psi|^2\psi, \quad (3.41)$$

with

$$\beta(t) = \frac{2}{3\sqrt{\frac{16}{81}\sin^2 t + \cos^2 t}}, \quad (3.42)$$

$$\varepsilon(t) = \frac{-2\delta(0)\sin t}{3\sqrt{\frac{16}{81}\sin^2 t + \cos^2 t}}. \quad (3.43)$$

(3.41) admits a bright-type soliton solution with absolute value of the form

$$|\psi_y(x, t)|^2 = \frac{\text{sech}^2(\beta(t)x + 2\gamma(t)y + \varepsilon(t))}{\sqrt{\beta^4(0) \sin^2 t + (2\alpha(0) \sin t + \cos t)^2}}. \quad (3.44)$$

We observe that  $\delta(0)$  produces a bending effect, see Figure 1.

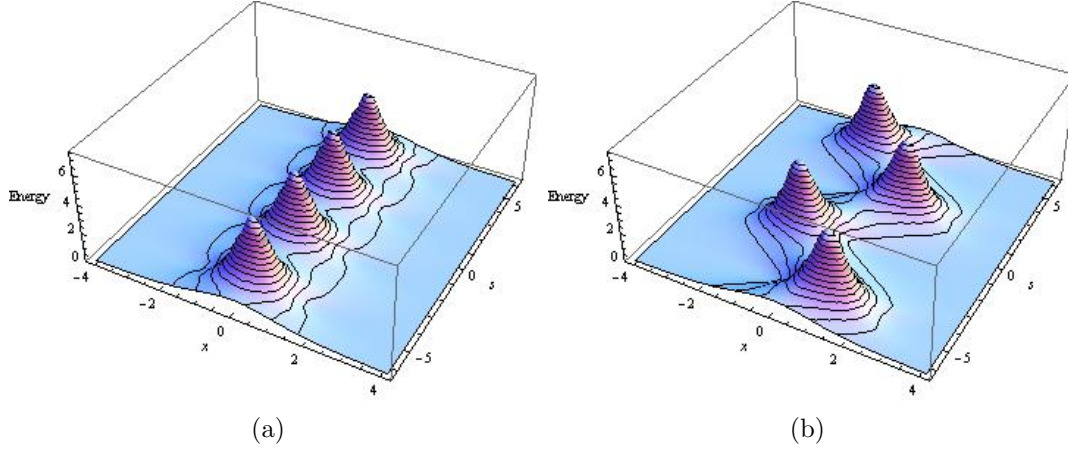


FIGURE 1. (a) Solution for (3.41) with  $\delta(0) = 0$ . (b) Solution for (3.41) with  $\delta(0) = 1$ .

**Example 7.** (*Dark-type soliton*) The equation (3.27) with  $\delta(0)$  a parameter,  $h(0) = 2$ ,  $\beta(0) = 2/3$ ,  $\mu(0) = 1$  and  $\alpha(0) = \gamma(0) = \varepsilon(0) = \kappa(0) = 0$  becomes

$$i\psi_t = -\frac{1}{2}\psi_{xx} + (1 - \beta^4)x^2\psi - 2\beta^3(t)\varepsilon(t)x\psi + \beta^2(t)\varepsilon^2(t)\psi + \frac{8}{9\sqrt{\frac{16}{81}\sin^2 t + \cos^2 t}}|\psi|^2\psi \quad (3.45)$$

and admits a dark-type soliton solution with absolute value of the form

$$|\psi_y(x, t)|^2 = \frac{\tanh^2(\beta(t)x + 2\gamma(t)y + \varepsilon(t))}{\sqrt{\beta^4(0) \sin^2 t + (2\alpha(0) \sin t + \cos t)^2}}. \quad (3.46)$$

We observe again that  $\delta(0)$  produces a bending effect, see Figure 2.

#### 4. DYNAMICS OF EXPLICIT SOLUTIONS THROUGH PARAMETERS

In this section using Lemmas 1 and 2, we will give examples of nonautonomous nonlinear Schrödinger equations with multiparameter solutions. These toy examples show the control of the dynamics of the solutions as an application of our multiparameter approach. We have prepared a Mathematica file as supplemental material for this Section. In this file all the solutions for this Section are verified. Also, all the formulas from the appendix have been verified previously in [29]. In the following two examples we use Lemma 2 from the appendix.

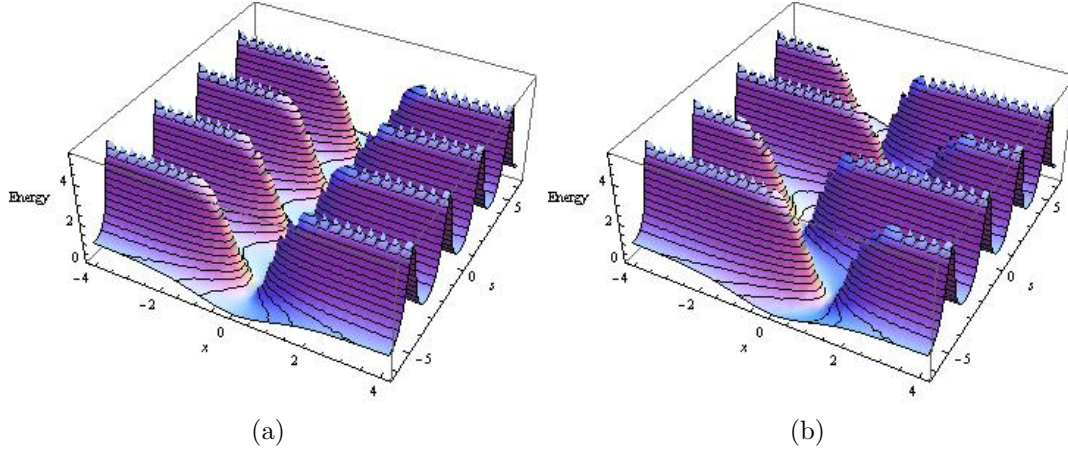


FIGURE 2. (a) Solution for (3.45) with  $\delta(0) = 0$ . (b) Solution for (3.45) with  $\delta(0) = 1$ .

**4.1. Dynamics of the bright soliton: Bending propagation.** Consider the nonautonomous nonlinear Schrödinger equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + ix \tanh t \psi_x + i \cosh t \psi + \frac{e^{2\sinh t} \operatorname{sech} t}{2 \sinh t + \cosh t} |\psi|^2 \psi = 0, \quad x \in \mathbb{R}, \quad t > 0. \quad (4.1)$$

Then, the characteristic equation and its solution are given by

$$\begin{aligned} \mu'' - (4 \cosh t - 2 \tanh t) \mu' + (4 \cosh^2 t - 6 \sinh t) \mu &= 0. \\ \mu(t) &= (2 + \coth t) e^{2 \sinh t} \tanh t. \end{aligned}$$

By Lemma 2 (4.1) can be reduced to:

$$iu_\tau - u_{\xi\xi} - 2|u|^2 u = 0, \quad \xi = \beta(t)x + \varepsilon(t), \quad \tau = \gamma(t), \quad (4.2)$$

where the general solution of the Riccati system associated is:

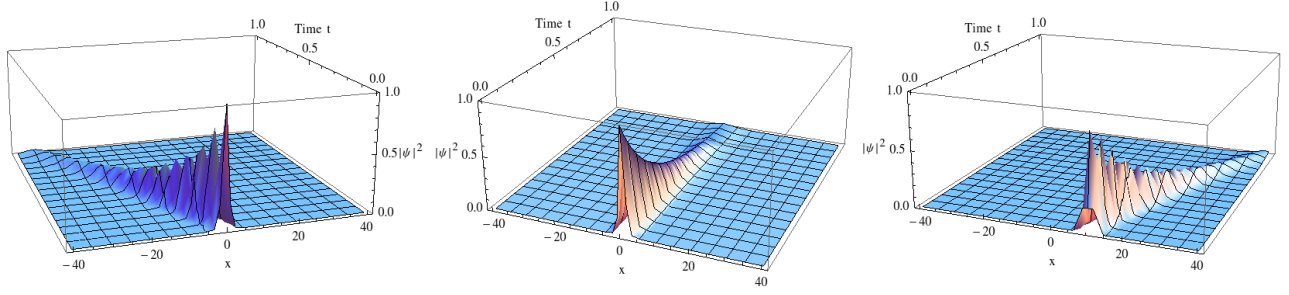
$$\begin{aligned} \alpha(t) &= \frac{\operatorname{csch} t \operatorname{sech} t}{2 + \coth t}, \quad \beta(t) = \frac{\operatorname{csch} t}{2 + \coth t}, \quad \gamma(t) = \gamma(0) - \frac{1}{4 + 2 \coth t}, \\ \delta(t) &= \frac{\delta(0) \operatorname{csch} t}{2 + \coth t}, \quad \varepsilon(t) = \varepsilon(0) - \frac{\delta(0)}{2 + \coth t}, \quad \kappa(t) = \kappa(0) - \frac{\delta(0)^2}{4 + 2 \coth t}. \end{aligned}$$

In order to use the similarity transformation method we proceed to use the familiar solution for (4.2):  $u(\tau, \xi) = \sqrt{v} \operatorname{sech}(\sqrt{v}\xi) \exp(-iv\tau)$ ,  $v > 0$ .

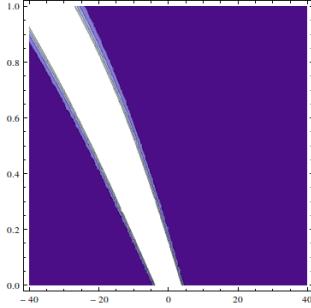
Therefore, a solution for the Schrödinger equation (4.1) is given by:

$$\begin{aligned} \psi(t, x) &= \sqrt{\frac{v \coth t}{2 + \coth t}} \operatorname{sech} \left[ \sqrt{v} \left( \frac{x \operatorname{csch} t - \delta(0)}{2 + \coth t} + \varepsilon(0) \right) \right] \\ &\times \exp \left[ i \left( \frac{2x^2 \operatorname{csch} t \operatorname{sech} t + 2\delta(0)x \operatorname{csch} t - \delta(0)^2 + v}{4 + 2 \coth t} \right) \right] \\ &\times \exp [i(\kappa(0) - v\gamma(0)) - \sinh t]. \end{aligned} \quad (4.3)$$

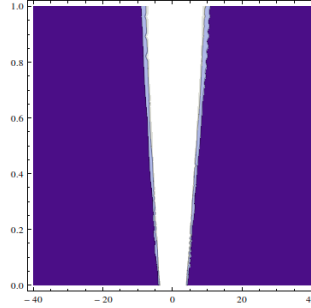
The dynamics of the solution (4.3) are shown in Figure 3, where it was possible to produce a change in the central axis of the bright soliton for certain values  $\delta(0)$  and  $\varepsilon(0)$ .



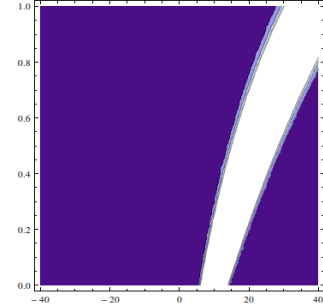
(a) Bright soliton solution bended to the left:  $v = 1$ ,  $\delta(0) = -30$  and  $\delta(0) = \frac{1}{6}$  and  $\varepsilon(0) = -\frac{1}{6}$ .  
 (b) Centered bright soliton:  $v = 1$ ,  $\delta(0) = 0$  and  $\varepsilon(0) = -10$ .  
 (c) Bright soliton solution bended to the right:  $v = 1$ ,  $\delta(0) = 0$  and  $\varepsilon(0) = -10$ .



(d) Contour of the bright soliton solution bended to the left.



(e) Contour for the centered bright soliton.



(f) Contour of the bright soliton solution bended to the right.

FIGURE 3. Control on the dynamics of the solution (4.3) for the equation (4.1).

**4.2. Dynamics of the dark soliton: Bending propagation.** Consider the nonautonomous nonlinear Schrödinger equation

$$i\psi_t = -\frac{1}{2} \cosh t \psi_{xx} + \frac{1}{2} \cosh t \psi(x^2 - i) - ix \cosh t \psi_x + \frac{4 \cosh t}{1 + \sinh t} |\psi|^2 \psi, \quad x \in \mathbb{R}, \quad t > 0; \quad (4.4)$$

then, the characteristic equation and its solution are respectively

$$\begin{aligned} \mu'' - \tanh t \mu' &= 0, \\ \mu(t) &= 1 + \sinh t. \end{aligned}$$

By Lemma 2 (4.4) can be reduced to

$$iu_\tau + u_{\xi\xi} - 2|u|^2 u = 0, \quad \xi = \beta(t)x + \varepsilon(t), \quad \tau = \gamma(t). \quad (4.5)$$

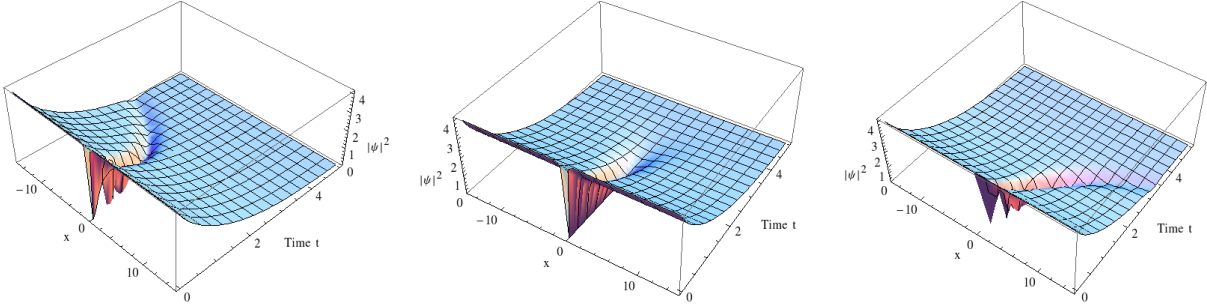
The general solution of the corresponding Riccati system is given by

$$\begin{aligned} \alpha(t) &= -\frac{1}{2 + 2 \cosh t}, \quad \beta(t) = \frac{1}{1 + \sinh t}, \quad \gamma(t) = \frac{2}{1 + \cosh t} - \gamma(0), \\ \delta(t) &= \frac{\delta(0)}{1 + \sinh t}, \quad \varepsilon(t) = \varepsilon(0) - \frac{2\delta(0)}{1 + \cosh t}, \quad \kappa(t) = \kappa(0) - \frac{\delta^2(0)}{2 + 2 \cosh t}. \end{aligned}$$

In order to use the similarity transformation method we proceed to use the familiar solution for (4.5):  $u(\tau, \xi) = A \tanh(A\xi) \exp(-2iA^2\tau)$ ,  $A \in \mathbb{R}$ . As a consequence, a solution for the Schrödinger equation (4.4) has the following form:

$$\begin{aligned} \psi(t, x) &= \frac{A}{\sqrt{1 + \sinh t}} \tanh \left[ A \left( \frac{2x \operatorname{csch} t - 2\delta(0)}{\operatorname{csch} t + 1} + \varepsilon(0) \right) \right] \\ &\times \exp \left[ i \left( \frac{-x^2 + 2\delta(0)x \operatorname{csch} t - \delta^2(0) - 8A^2}{2 + 2 \operatorname{csch} t} \right) \right] \\ &\times \exp \left[ i \left( \kappa(0) + 2A^2\gamma(0) \right) \right]. \end{aligned} \quad (4.6)$$

Figure 4 describes the evolution in time of the solution (4.6). Again for this case we have bending propagation.



(a) Dark soliton solution bended to the left:  $A = 2$ ,  $\delta(0) = -1$  and  $\varepsilon(0) = 0$ .  
 (b) Centered dark soliton:  $A = 2$ ,  $\delta(0) = 0$ ,  $\varepsilon(0) = 0$ .  
 (c) Dark soliton solution bended to the right:  $A = 2$ ,  $\delta(0) = 0$  and  $\varepsilon(0) = -2$ .

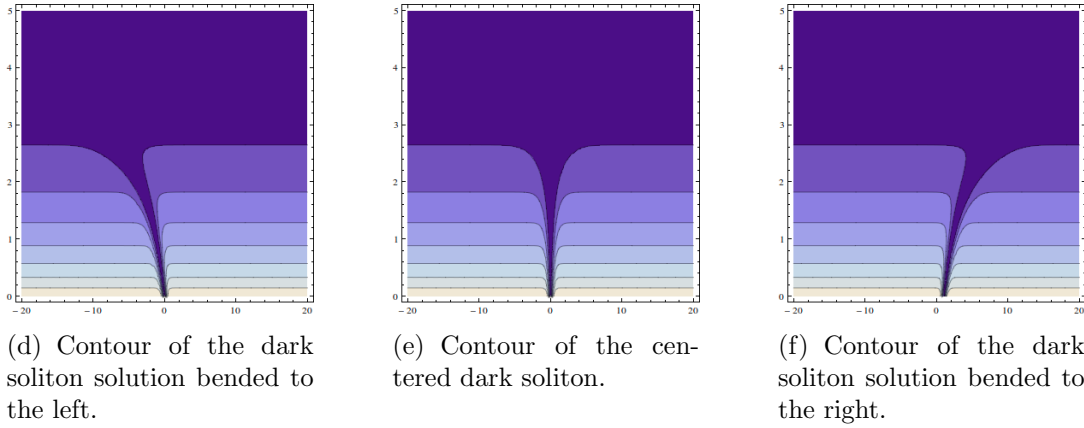


FIGURE 4. Control on the dynamics of the solution (4.6) for the equation (4.4).

The difficulty of applying Lemma 2 is solving the Riccati system. Next, we present an alternative approach to deal with the Riccati system and see how the dynamics of the solutions change with multiparameters.

**4.3. An alternative method to solve the coupled Riccati system and applications to soliton solutions.** Assume the following conditions on the Riccati system (6.1)-(6.6):  $a(t) = -l_0$  with  $l_0 = \pm 1$ ,  $\beta(t) = 1$ ,  $\tau(t) = t$  and  $\varepsilon(t) = 0$ . Under this hypothesis one obtains the explicit

formulas  $\alpha(t) = l_0 c(t)/4$ ,  $\delta(t) = -l_0 g(t)/2$  and  $h(t) = -l_0 \lambda \mu(t)$ , where the last expression shows the unique dependence of the coefficient of the nonlinearity in terms of the characteristic function  $\mu$ . Furthermore one obtains the particular Riccati system

$$\frac{dc}{dt} + c^2 + 4l_0 b = 0, \quad (4.7)$$

$$\frac{dg}{dt} + 2l_0 f + cg = 0, \quad (4.8)$$

$$\frac{d\kappa}{dt} + \frac{l_0}{4} g^2 = 0, \quad (4.9)$$

$$\frac{d\mu}{dt} = (2d - c)\mu. \quad (4.10)$$

The solution of this system is given by

$$\frac{dc}{dt} + c^2 + 4l_0 b = 0, \quad (4.11)$$

$$\alpha(t) = l_0 \frac{c(t)}{4}, \quad \delta(t) = -l_0 \frac{g(t)}{2}, \quad h(t) = -l_0 \lambda \mu(t), \quad (4.12)$$

$$\kappa(t) = \kappa(0) - \frac{l_0}{4} \int_0^t g^2(z) dz, \quad (4.13)$$

$$\mu(t) = \mu(0) \exp \left( \int_0^t (2d(z) - c(z)) dz \right), \quad \mu(0) \neq 0, \quad (4.14)$$

$$g(t) = g(0) - 2l_0 \exp \left( - \int_0^t c(z) dz \right) \int_0^t \exp \left( \int_0^z c(y) dy \right) f(z) dz. \quad (4.15)$$

Further, with these restrictions we have a way to construct transformations that allow us to prove uniqueness of the solutions.

**Lemma 1.** *Suppose that  $h(t) = -l_0 \lambda \mu(t)$  with  $\lambda \in \mathbb{R}$ ,  $l_0 = \pm 1$  and that  $c(t)$ ,  $\alpha(t)$ ,  $\delta(t)$ ,  $\kappa(t)$ ,  $\mu(t)$  and  $g(t)$  satisfy the equations (4.11)-(4.15). Then*

$$\psi(t, x) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} u(t, x) \quad (4.16)$$

*is a solution to the Cauchy problem for the nonautonomous nonlinear Schrödinger equation*

$$i\psi_t = l_0 \psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x + h(t)|\psi|^2\psi, \quad (4.17)$$

$$\psi(0, x) = \psi_0(x) \quad (4.18)$$

*if and only if  $u(t, x)$  is a solution of the Cauchy problem for the standard nonlinear Schrödinger equation*

$$iu_t - l_0 u_{xx} + l_0 \lambda |u|^2 u = 0. \quad (4.19)$$

$$u(0, x) = \sqrt{\mu(0)} e^{-i(\alpha(0)x^2 + \delta(0)x + \kappa(0))} \psi_0(x). \quad (4.20)$$

The following proposition establishes the uniqueness of classical solutions for the nonautonomous nonlinear Schrödinger equation (4.17).



**Proposition 1.** *Assume that equations (4.11)-(4.15) are satisfied (corresponding to  $l_0 = -1$ ). If  $c(t)$ ,  $d(t)$  and  $f(t) \in C^1([-T, T])$  for some  $T > 0$  and  $h(t) = \lambda\mu(t)$  with  $\lambda \in \mathbb{R}$ , then the Cauchy problem for the nonlinear nonautonomous Schrödinger equation*

$$i\psi_t = -\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x + h(t)|\psi|^2\psi \quad (4.21)$$

$$\psi(0, x) = \psi_0(x) \quad (4.22)$$

has a unique classical solution in the space  $L_t^\infty L_x^q([-T, T] \times \mathbb{R})$  for  $q = 2, \infty$ .

*Proof.* Let's consider  $\psi^1, \psi^2 \in C_{t,x}^2([-T, T] \times \mathbb{R})$  classical solutions for the Cauchy problem (4.21)-(4.22) in the space  $L_t^\infty L_x^q([-T, T] \times \mathbb{R})$  for  $q = 2, \infty$ . By Lemma 1 and the conditions in the coefficients  $c(t)$ ,  $d(t)$ ,  $f(t)$  and  $h(t)$ , we have

$$u^j(t, x) = \sqrt{\mu(t)} e^{-i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} \psi^j(t, x) \in C_{t,x}^2([-T, T] \times \mathbb{R}) \quad \text{with } j = 1, 2, \quad (4.23)$$

which are classical solutions for the Cauchy problem (4.19)-(4.20) on  $[-T, T]$ , and initial condition

$$u^j(0, x) = \sqrt{\mu(0)} e^{-i(\alpha(0)x^2 + \delta(0)x + \kappa(0))} \psi_0(x) \quad \text{for } j = 1, 2.$$

Therefore, for each  $j = 1, 2$  we have

$$\|u^j\|_{L_t^\infty L_x^q([-T, T] \times \mathbb{R})} \leq M \|\psi^j\|_{L_t^\infty L_x^q([-T, T] \times \mathbb{R})}, \quad q = 2, \infty,$$

where  $M > 0$  is the maximum for  $\mu(t)$  in the interval  $[-T, T]$ . Using the classical uniqueness result given in [60], we have  $u^1 = u^2$ , and so, the final result is obtained by multiplying equation (4.23) by the factor  $e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} / \sqrt{\mu(t)}$ .  $\square$

Now we want to see how the dynamics of the solutions of the nonlinear Schrödinger equation (4.17) change when the parameters of dissipation,  $d(t)$ , and the nonlinear term,  $h(t)$  change. We will use the alternative Riccati system (4.7)-(4.10) and therefore Lemma 1.

**4.4. Perturbations of the bright soliton: Competition between dissipation and nonlinearity.** Let's consider the nonautonomous nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} + \frac{x^2}{4} (\sin^2 t - \cos t) \psi + ix \sin t \psi_x - id(t)\psi + h(t)|\psi|^2\psi, \quad t, x \in \mathbb{R}. \quad (4.24)$$

Then the functions  $\alpha$ ,  $\delta$  and  $\kappa$  are respectively:

$$\alpha(t) = \frac{\sin t}{4}, \quad \delta(t) = 0, \quad \kappa(t) = \kappa(0).$$

We will construct explicit solutions for (4.24) using Lemma 1 and one of the solutions for the NLS equation

$$iu_t + u_{xx} + 3|u|^2u = 0, \quad t, x \in \mathbb{R},$$

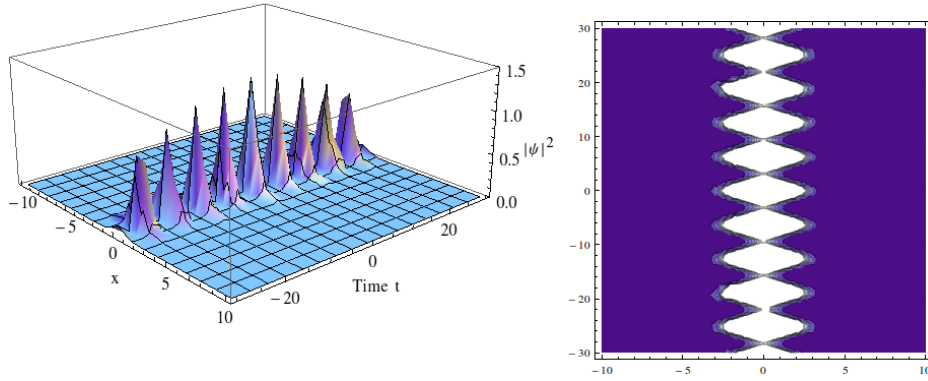
that is given by

$$u(t, x) = \sqrt{-\frac{2v}{3}} \operatorname{sech}(\sqrt{-v}x) \exp(-vit), \quad v < 0.$$

4.4.1. *Periodic Solutions for (4.24) with  $d(t) = \sin t$  and  $h(t) = -3e^{3-3\cos t}$ .* In this case one obtains a solution of the following form:

$$\begin{aligned} \psi(t, x) &= \exp \left[ \frac{3}{2} (\cos t - 1) + i \left( \frac{x^2}{4} \sin t + \kappa(0) - vt \right) \right] \\ &\times \sqrt{-\frac{2v}{3}} \operatorname{sech}(\sqrt{-v}x), \end{aligned} \quad (4.25)$$

see Figure 5.



(a) Solution with values  $v = -2$  and  $\kappa(0) = 0$ . (b) Contour of the solution.

FIGURE 5. Dynamics of the solution (4.25) for equation (4.24) with  $d(t) = \sin t$  and  $h(t) = -3e^{3-3\cos t}$ .

4.4.2. *Solutions with fast decay (4.24) with  $d(t) = (4t - \sin t)/2$  and  $h(t) = -3e^{2t^2}$ .* The choice of the parameters allows us to construct a solution with fast decay for large values of time:

$$\psi(t, x) = \sqrt{-\frac{2v}{3}} \operatorname{sech}(\sqrt{-v}x) \exp \left[ i \left( \frac{x^2}{4} \sin t + \kappa(0) - vt \right) - t^2 \right], \quad (4.26)$$

see Figure 6.

**4.5. Perturbations of the Peregrine soliton.** We are interested to see how the parameters change the dynamics of solutions for (4.17), and for this end we consider

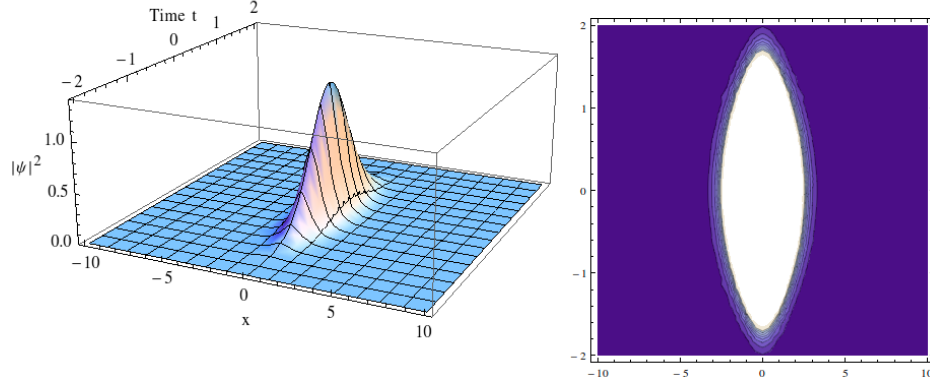
$$i\psi_t = -\psi_{xx} + x^2 (t^2 - 1/2) \psi + 2it\psi_x(x + e^{t^2}) - id(t)\psi - e^{t^2}x\psi + h(t)|\psi|^2\psi, \quad t, x \in \mathbb{R}. \quad (4.27)$$

As before we can find explicitly

$$\alpha(t) = t/2, \quad \delta(t) = te^{t^2}, \quad \kappa(t) = \kappa(0) + e^{2t^2} \left( 2t - \sqrt{2}D(\sqrt{2}t) \right) / 8,$$

where  $D(t) = e^{-t^2} \int_0^t e^{z^2} dz$  is the Dawson function. We will construct explicit solutions for (4.27) using Lemma 2 to reduce it to

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad t, x \in \mathbb{R}, \quad (4.28)$$



(a) Solution with values  $v = -2$  and  $\kappa(0) = 0$ . (b) Contour of the solution.

FIGURE 6. Dynamics of the solution (4.26) for equation (4.24) with  $d(t) = (4t - \sin t)/2$  and  $h(t) = -3e^{2t^2}$ .

and one of the solutions for (4.28) is

$$u(t, x) = A \exp(2iA^2t) \left( \frac{3 + 16iA^2t - 16A^4t^2 - 4A^2x^2}{1 + 16A^4t^2 + 4A^2x^2} \right), \quad A \in \mathbb{R}.$$

4.5.1. *Peregrine-type soliton for (4.27) with  $d(t) = \tanh t - t$  and  $h(t) = -8 \cosh^2 t$ .* The correct choice of the parameters  $d(t)$  and  $h(t)$  allows us to construct solutions with properties similar to those of the classical Peregrine soliton, as can be seen in Figure 7. The solution for this case will be given by

$$\begin{aligned} \psi(t, x) &= \exp \left[ i \left( \frac{t}{2} x^2 + t e^{t^2} x + \kappa(0) + \frac{1}{8} e^{2t^2} (2t - \sqrt{2} D(\sqrt{2}t)) \right) \right] \\ &\times \frac{A}{2} \exp(2A^2t) \left( \frac{3 + 16iA^2t - 16A^4t^2 - 4A^2x^2}{1 + 16A^4t^2 + 4A^2x^2} \right) \operatorname{sech} t. \end{aligned} \quad (4.29)$$

4.5.2. *Dynamics of the solution for (4.27) with  $d(t) = -(\sin 2t + t)$  and  $h(t) = -2e^{-2 \sin^2 t}$ .* Here we see how the solutions are perturbed, see Figure 8 in the appendix. The solution is given by

$$\begin{aligned} \psi(t, x) &= A \exp \left[ i \left( \frac{t}{2} x^2 + t e^{t^2} x + \kappa(0) + \frac{1}{8} e^{2t^2} (2t - \sqrt{2} D(\sqrt{2}t)) \right) \right] \\ &\times \left( \frac{3 + 16iA^2t - 16A^4t^2 - 4A^2x^2}{1 + 16A^4t^2 + 4A^2x^2} \right) \exp(2iA^2t + \sin^2 t). \end{aligned} \quad (4.30)$$

## 5. FINAL REMARKS

In this work, inspired by the work of Mahric on multiparameter solutions for the linear Schrödinger equation with quadratic potential, we have established a relationship between solutions with parameters of Riccati-Ermakov systems with the dynamics of nonlinear Schrödinger equations with

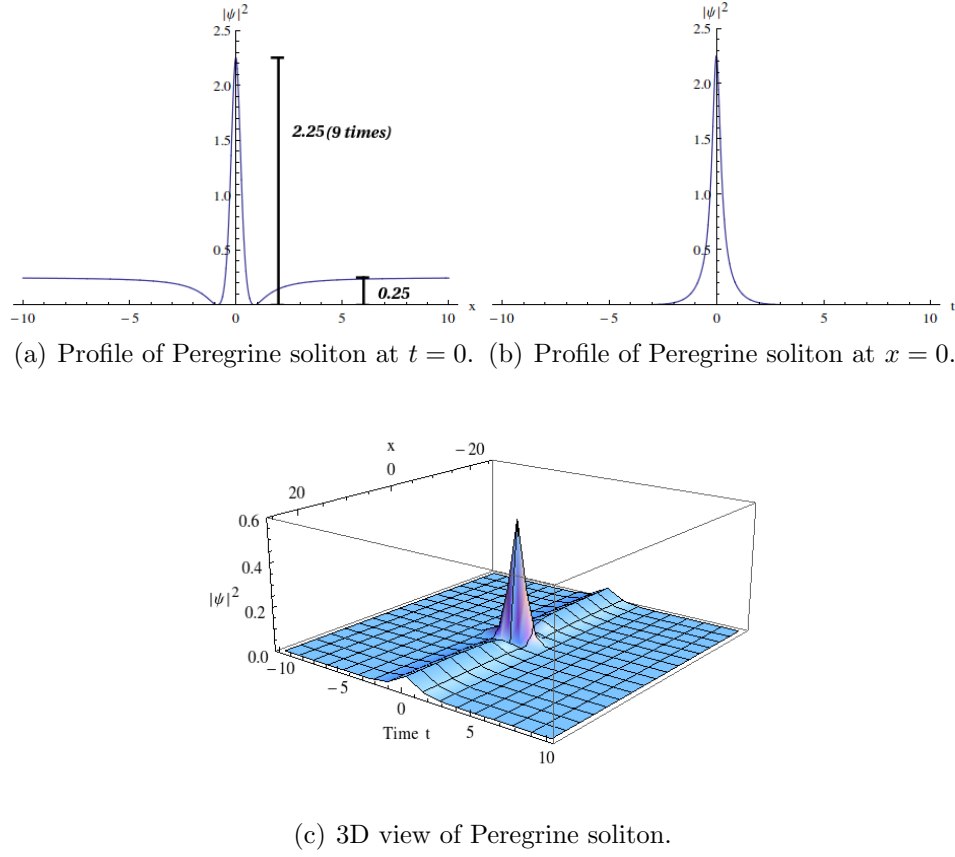


FIGURE 7. Peregrine soliton solution for (4.29) with  $A = 0.5$  and  $\kappa(0) = 0$  for (4.27) with  $d(t) = \tanh(t) - t$  and  $h(t) = -8 \cosh^2(t)$ .

variable coefficients of the form (1.2). We have shown that for special coefficients of (1.2) it is possible to find explicit solutions presenting blow up, periodic soliton solutions with bending properties and more. This work should motivate further analytical and numerical studies looking to clarify the connections on the dynamics of variable-coefficient NLS with the dynamics of Riccati-Ermakov systems.

**Acknowledgement 1.** *The second named author was supported by the Simons Foundation #316295. The authors thanks Drs. A. Mahalov, S. Roudenko and S. K. Suslov for many different valuable discussions of this project and valuable mentorship.*

## 6. APPENDIX: NONLINEAR COUPLED RICCATI-ERMAKOV SYSTEMS AND SIMILARITY TRANSFORMATIONS

All the formulas from this appendix have been verified previously in [29].

**6.1. Modified Riccati system and a similarity transformation.** In section 4 we need a slightly modified nonlinear coupled Riccati system that includes for convenience of our results a parameter

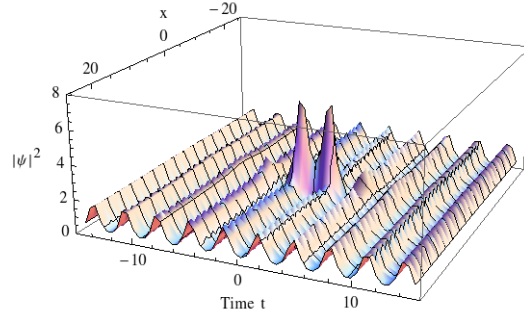
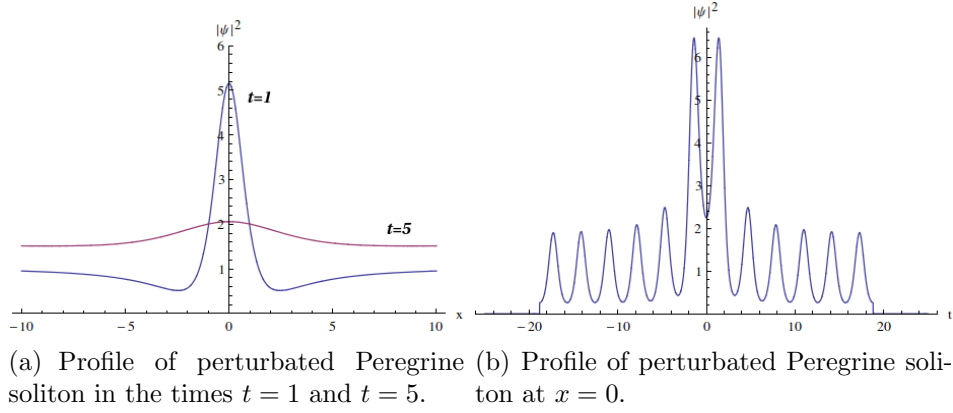


FIGURE 8. Dynamics of perturbed Peregrine soliton solution (4.30) with  $A = 0.5$  and  $\kappa(0) = 0$  for (4.27) with  $d(t) = -(\sin(2t) - t)$  and  $h(t) = -2e^{-2\sin^2(t)}$ .

$l_0 = \pm 1$  (the case  $l_0 = 1$  has already been considered in [13], [54], [58]):

$$\frac{d\alpha}{dt} + b(t) + 2c(t)\alpha + 4a(t)\alpha^2 = 0, \quad (6.1)$$

$$\frac{d\beta}{dt} + (c(t) + 4a(t)\alpha(t))\beta = 0, \quad (6.2)$$

$$\frac{d\gamma}{dt} + l_0 a(t)\beta^2(t) = 0, \quad l_0 = \pm 1, \quad (6.3)$$

$$\frac{d\delta}{dt} + (c(t) + 4a(t)\alpha(t))\delta = f(t) + 2\alpha(t)g(t), \quad (6.4)$$

$$\frac{d\varepsilon}{dt} = (g(t) - 2a(t)\delta(t))\beta(t), \quad (6.5)$$

$$\frac{d\kappa}{dt} = g(t)\delta(t) - a(t)\delta^2(t). \quad (6.6)$$

Considering the standard substitution

$$\alpha = \frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \quad (6.7)$$

it follows that the Riccati equation (6.1) becomes

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0, \quad (6.8)$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right). \quad (6.9)$$

We will refer to (6.8) as the characteristic equation of the Riccati system. Here  $a(t)$ ,  $b(t)$ ,  $c(t)$ ,  $d(t)$ ,  $f(t)$  and  $g(t)$  are real value functions depending only on the variable  $t$ . A solution of the Riccati system (6.1)-(6.6) with multiparameters is given by the following expressions (with the respective inclusion of the parameter  $l_0$ ) [13], [54], [58]:

$$\mu(t) = 2\mu(0)\mu_0(t)(\alpha(0) + \gamma_0(t)), \quad (6.10)$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))}, \quad (6.11)$$

$$\beta(t) = -\frac{\beta(0)\beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0)\mu(0)}{\mu(t)}w(t), \quad (6.12)$$

$$\gamma(t) = l_0\gamma(0) - \frac{l_0\beta^2(0)}{4(\alpha(0) + \gamma_0(t))}, \quad l_0 = \pm 1, \quad (6.13)$$

$$\delta(t) = \delta_0(t) - \frac{\beta_0(t)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (6.14)$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (6.15)$$

$$\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))}, \quad (6.16)$$

subject to the initial arbitrary conditions  $\mu(0)$ ,  $\alpha(0)$ ,  $\beta(0) \neq 0$ ,  $\gamma(0)$ ,  $\delta(0)$ ,  $\varepsilon(0)$  and  $\kappa(0)$ .  $\alpha_0$ ,  $\beta_0$ ,  $\gamma_0$ ,  $\delta_0$ ,  $\varepsilon_0$  and  $\kappa_0$  are given explicitly by

$$\alpha_0(t) = \frac{1}{4a(t)} \frac{\mu'_0(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \quad (6.17)$$

$$\beta_0(t) = -\frac{w(t)}{\mu_0(t)}, \quad w(t) = \exp\left(-\int_0^t (c(s) - 2d(s)) ds\right), \quad (6.18)$$

$$\gamma_0(t) = \frac{d(0)}{2a(0)} + \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)}, \quad (6.19)$$

$$\delta_0(t) = \frac{w(t)}{\mu_0(t)} \int_0^t \left[ \left( f(s) - \frac{d(s)}{a(s)} g(s) \right) \mu_0(s) + \frac{g(s)}{2a(s)} \mu'_0(s) \right] \frac{ds}{w(s)}, \quad (6.20)$$

$$\begin{aligned} \varepsilon_0(t) = & -\frac{2a(t)w(t)}{\mu'_0(t)} \delta_0(t) + 8 \int_0^t \frac{a(s)\sigma(s)w(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s)) ds \\ & + 2 \int_0^t \frac{a(s)w(s)}{\mu'_0(s)} \left[ f(s) - \frac{d(s)}{a(s)} g(s) \right] ds, \end{aligned} \quad (6.21)$$

$$\kappa_0(t) = \frac{a(t)\mu_0(t)}{\mu'_0(t)} \delta_0^2(t) - 4 \int_0^t \frac{a(s)\sigma(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s))^2 ds \quad (6.22)$$

$$-2 \int_0^t \frac{a(s)}{\mu'_0(s)} (\mu_0(s) \delta_0(s)) \left[ f(s) - \frac{d(s)}{a(s)} g(s) \right] ds,$$

with  $\delta_0(0) = g_0(0) / (2a(0))$ ,  $\varepsilon_0(0) = -\delta_0(0)$ ,  $\kappa_0(0) = 0$ . Here  $\mu_0$  and  $\mu_1$  represent the fundamental solution of the characteristic equation subject to the initial conditions  $\mu_0(0) = 0$ ,  $\mu'_0(0) = 2a(0) \neq 0$  and  $\mu_1(0) \neq 0$ ,  $\mu'_1(0) = 0$ .

Using the system (6.11)-(6.16), in [58] we see a generalized lens transformation is presented. Next we recall this result (here we present a slight perturbation introducing the parameter  $l_0 = \pm 1$  in order to use Peregrine-type soliton solutions):

**Lemma 2** ( $l_0 = 1$ , [58]). *Assume that  $h(t) = \lambda a(t) \beta^2(t) \mu(t)$  with  $\lambda \in \mathbb{R}$ . Then the substitution*

$$\psi(t, x) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} u(\tau, \xi), \quad (6.23)$$

where  $\xi = \beta(t)x + \varepsilon(t)$  and  $\tau = \gamma(t)$ , transforms the equation

$$i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x + h(t)|\psi|^2\psi$$

into the standard Schrödinger equation

$$iu_\tau - l_0 u_{\xi\xi} + l_0 \lambda |u|^2 u = 0, \quad l_0 = \pm 1, \quad (6.24)$$

as long as  $\alpha, \beta, \gamma, \delta, \varepsilon$  and  $\kappa$  satisfy the Riccati system (6.1)-(6.6) and also equation (6.7).

**6.2. Ermakov System and a Similarity Transformation.** We recall the following useful results for sections 2 and 3.

**Lemma 3** ([33], [36] and [37]). *The following nonlinear coupled system (Ermakov system)*

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = c_0 a \beta^4, \quad (6.25)$$

$$\frac{d\beta}{dt} + (c + 4a\alpha) \beta = 0, \quad (6.26)$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0, \quad (6.27)$$

$$\frac{d\delta}{dt} + (c + 4a\alpha) \delta = f + 2cg + 2c_0 a \beta^3 \varepsilon, \quad (6.28)$$

$$\frac{d\varepsilon}{dt} = (g - 2a\delta) \beta, \quad (6.29)$$

$$\frac{d\kappa}{dt} = g\delta - a\delta^2 + c_0 a \beta^2 \varepsilon^2 \quad (6.30)$$

$$\alpha(t) = -\frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)} \quad (6.31)$$

admits the following multiparameter solution given explicitly by

$$\mu(t) = \mu_0(t) \mu(0) \sqrt{4(\gamma_0(t) + \alpha(0))^2 + \beta^4(0)}, \quad (6.32)$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t) (\gamma_0(t) + \alpha(0))}{4(\gamma_0(t) + \alpha(0))^2 + \beta^4(0)}, \quad (6.33)$$

$$\beta(t) = -\frac{\beta(0)\beta_0(t)}{\sqrt{4(\gamma_0(t) + \alpha(0))^2 - \beta^4(0)}}, \quad (6.34)$$

$$\gamma(t) = \gamma(0) - \frac{1}{2} \arctan \frac{\beta^2(0)}{2(\gamma_0(t) + \alpha(0))} \quad (6.35)$$

and

$$\delta(t) = \delta_0(t) - \beta_0(t) \frac{\varepsilon(0)\beta^3(0) + 2(\gamma_0(t) + \alpha(0))(\varepsilon_0(t) + \delta(0))}{4(\gamma_0(t) + \alpha(0))^2 + \beta^4(0)} \quad (6.36)$$

$$\varepsilon(t) = \frac{-\beta(0)(\delta(0) + \varepsilon_0(t)) + 2\varepsilon(0)(\gamma_0(t) + \alpha(0))}{\sqrt{4(\gamma_0(t) + \alpha(0))^2 + \beta^4(0)}}, \quad (6.37)$$

$$\begin{aligned} \kappa(t) = \kappa_0(t) + \kappa(0) - \frac{\beta^3(0)\varepsilon(0)(\varepsilon_0(t) + \delta(0))}{4(\gamma_0(t) + \alpha(0))^2 + \beta^4(0)} \\ + \frac{(\gamma_0(t) + \alpha(0))[\beta^2(0)\varepsilon^2(0) - (\varepsilon_0(t) + \delta(0))^2]}{4(\gamma_0(t) + \alpha(0))^2 + \beta^4(0)}, \end{aligned} \quad (6.38)$$

subject to arbitrary initial data  $\mu(0)$ ,  $\alpha(0)$ ,  $\beta(0) \neq 0$ ,  $\gamma(0)$ ,  $\delta(0)$ ,  $\varepsilon(0)$ ,  $\kappa(0)$  where  $\alpha_0(t)$ ,  $\beta_0(t)$ ,  $\gamma_0(t)$ ,  $\delta_0(t)$ ,  $\varepsilon_0(t)$  and  $\kappa_0(t)$  are given by (6.17)-(6.22).

We will also need a 2D version of the results above for the blow-up results of Section 2:

**Lemma 4.** ([39]) *The nonlinear equation*

$$\begin{aligned} i\psi_t = -a(\psi_{xx} + \psi_{yy}) + b(x^2 + y^2)\psi - ic(x\psi_x + y\psi_y) - 2id\psi \\ - (xf_1 + yf_2)\psi + i(g_1\psi_x + g_2\psi_y) + h|\psi|^{2s}\psi, \end{aligned} \quad (6.39)$$

where  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $f_{1,2}$  and  $g_{1,2}$  are real-valued functions of  $t$ , can be transformed to

$$i\chi_\tau - l_0(\chi_{\xi\xi} + \chi_{\eta\eta}) = -l_0h_0|\chi|^{2s}\chi \quad (l_0 = \pm 1) \quad (6.40)$$

by the ansatz

$$\psi = \mu^{-1} e^{i(\alpha(x^2+y^2) + (\delta_1x + \delta_2y) + \kappa_1 + \kappa_2)} \chi(\xi, \eta, \tau), \quad (6.41)$$

where  $\xi = \beta(t)x + \varepsilon_1(t)$ ,  $\eta = \beta(t)y + \varepsilon_2(t)$ ,  $\tau = \gamma(t)$ ,  $h(t) = h_0a(t)\beta^2(t)\mu^{2s}(t)$  ( $h_0$  is a constant), provided that

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = 0, \quad (6.42)$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0, \quad (6.43)$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0, \quad (6.44)$$

$$\frac{d\delta_{1,2}}{dt} + (c + 4a\alpha)\delta_{1,2} = f_{1,2} + 2g\alpha, \quad (6.45)$$

$$\frac{d\varepsilon_{1,2}}{dt} = (g - 2a\delta_{1,2})\beta, \quad (6.46)$$

$$\frac{d\kappa_{1,2}}{dt} = g\delta_{1,2} - a\delta_{1,2}^2. \quad (6.47)$$



Here,

$$\alpha = \frac{1}{4a} \frac{\mu'}{\mu} - \frac{d}{2a}, \quad (6.48)$$

and solutions of the system (6.42)–(6.47) are given by (6.10)–(6.16).

## REFERENCES

- [1] M. Ablowitz and P. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge Univ. Press. (1991).
- [2] M. Ablowitz and T. Hooroka, *Resonant intrachannel pulse interactions in dispersion-managed transmission systems*, IEEE Journal on Selected Topics in Quantum Electronics, 8 (2002) 603-615.
- [3] P. B. Acosta and E. Suazo, *Liouvillian propagators, Riccati equation and differential Galois theory*, J. Phys. A: Math. Theor. 46 (2013) 455203.
- [4] G. P. Agrawal, *Nonlinear Fiber Optics*, Academic Press, New York. Fourth Edition. 2007.
- [5] U. Al Khawaja, *A comparative analysis of Painlevé, Lax Pair and similarity transformation methods in obtaining the integrability conditions of nonlinear Schrödinger equations*, Journal of Physical Mathematics. 51 (2010).
- [6] V. Aldaya, F. Cossío, J. Guerrero and FF López-Ruiz, *The quantum Arnold transformation*, Journal of Physics A: Mathematical and Theoretical 44 (6), 065302.
- [7] R. Atre, P. K. Panigrahi and G. S. Agarwal, *Class of solitary waves solutions of the one-dimensional Gross-Pitaevskii equation*, Physical Review E 73 (2006) 056611.
- [8] T. Brugarino and M. Sciacca, *Integrability of an inhomogeneous nonlinear Schrödinger equation in Bose-Einstein condensates and fiber optics*, J. Math. Phys. 51 (2010).
- [9] R. Carles, *Nonlinear Schrödinger equation with time dependent potential*. Commun. Math. Sci. 9 (2011), no. 4, 937-964.
- [10] T. Cazenave and A. Haraux, *An Introduction to semilinear evolution equations*, Oxford Lecture Series in Mathematics and Its Applications, Oxford Science Publications, Clarendon Press, Oxford, 13, 1998.
- [11] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.
- [12] H-M. Chen and C.S. Liu, *Solitons in nonuniform media*, Phys. Rev. Lett. 37 (1976), 693 - 697.
- [13] R. Cordero-Soto, R. M. Lopez, E. Suazo, and S. K. Suslov, *Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields*, Lett. Math. Phys. 84 (2008) #2–3, 159–178.
- [14] R. Cordero-Soto and S. K. Suslov, *The degenerate parametric oscillator and Ince's equation*, Journal of Physics A: Mathematical and Theoretical. 44 (2011) 015101.
- [15] C-Q. Dai and Y-Y. Wang, *Infinite generation of soliton-like solutions for complex nonlinear evolution differential equations via the NLSE-based constructive method*, Appl. Math. Comput. 236 (2014) 606-612.
- [16] L. Debnath, *Nonlinear Partial Differential Equations: for Scientists and Engineers*, Third Edition, Birkhäuser, 2012.
- [17] V. V. Dodonov and V. I. Man'ko, *Coherent states and the resonance of a quantum damped oscillator*, Phys. Rev. A 20 (1979) # 2, 550-560.
- [18] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.
- [19] G. Fibich, *The nonlinear Schrödinger equation, singular solutions and optical collapse*, Springer 2015.
- [20] G. Fibich, *Self-focusing in the damped nonlinear Schrödinger equation*, SIAM J. Appl. Math., 61 (2001), 1680-1705.
- [21] G. Fibich, *Singular solutions of the subcritical nonlinear Schrödinger equation*, Physica D, 240 (2011), 1119-1122.
- [22] V. A. Fock, *Electromagnetic diffraction and propagation problems*, Pergamon Press, London, 1965.
- [23] X. G. He, D. Zhao, L. Li and H. G. Luo, *Engineering integrable nonautonomous nonlinear Schrödinger equations*, Physical Review E. 79, 056610 (2009).
- [24] J. He and Y. Li, *Designable integrability of the variable coefficient nonlinear Schrödinger equations*, Studies in applied mathematics, 126 (2010), 1-15.
- [25] J. S. He, E. G. Charalampidis, P. G. Kevrekidis and D. J. Frantzeskakis, *Rogue waves in nonlinear Schrödinger models with variable coefficients: Application to Bose-Einstein condensates*, Physics Letters A, 378 (2014), 577-583.
- [26] B. R. Holstein, *The linear potential propagator*, Am. J. Phys. 65 (1997) #5, 414–418.

- [27] T. Kato, *On nonlinear Schrödinger equations*, Ann. Ins. H. Poincaré Phys. Theor., 46 (1987), 113-129.
- [28] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, *Emergent nonlinear phenomena in Bose-Einstein condensates: Theory and experiment*. Springer Series of Atomic, Optical and Plasma Physics, Vol. 45 (2008).
- [29] C. Koutschan, E. Suazo and S. K. Suslov, *Fundamental laser modes in paraxial optics: from computer algebra and simulations to experimental observation*, Applied Physics B 121 (3) (2015), 315-336.
- [30] V. I. Kruglov, A. C. Peacock and J. D. Harvey, *Exact solutions of the generalized nonlinear Schrödinger equation with distributed coefficients*, Phys. Rev. E 71, 056619 (2005).
- [31] M. K. Kwong, *Uniqueness of positive solutions of  $\Delta u - u + u^p = 0$  in  $R^n$* . Arch. Ration. Mech. Anal. 105, 1989, 243-266.
- [32] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory*, Pergamon Press, Oxford, 1977.
- [33] N. Lanfear, R. M. López and S. K. Suslov, *Exact wave functions for generalized harmonic oscillators*, J. Russ. Laser Res. 32 (2011) # 4, 352-361.
- [34] P. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Commun. Pure. Appl. Math. 21 (1968).
- [35] F. Linares and G. Vega, *Introduction to Nonlinear Dispersive Equations*, Universitext series, Springer, 2009.
- [36] R. M. López; S. K. Suslov; J. M. Vega-Guzmán, *On a hidden symmetry of quantum harmonic oscillators*, J. Differ. Eqns. Appl. 19 (2013) # 4, 543-554.
- [37] R. M. López, S. K. Suslov, and J. M. Vega-Guzmán, *Reconstructing the Schrödinger groups*, Physica Scripta. 87 (2013) 038112.
- [38] A. Mahalov, E. Suazo and S. K. Suslov, *Spiral laser beams in inhomogeneous media*, Optics Letters, Vol. 38, (2013) Issue 15, 2763-2766.
- [39] A. Mahalov and S. K. Suslov, *Solution of paraxial wave equation for inhomogeneous media in linear and quadratic approximation*, Proc. Amer. Math. Soc. 143 (2015), 595-610.
- [40] M. E. Marhic, *Oscillating Hermite-Gaussian wave functions of the harmonic oscillator*, Lett. Nuovo Cim. 22 (1978) # 8, 376-378.
- [41] V. G. Marikhin, A. B. Shabat, M. Boiti, F. Pempinelli, *Self-similar solutions of equations of the nonlinear Schrödinger type*, J. Exp. Theor. Phys., vol. 90, (2000), 553-561.
- [42] S. Medvedev, O. Shtyrina, S. Musher and M. Fedoruk, *Path-averaged optical soliton in double-periodic dispersion-managed systems*, Phys. Rev. E 66, 066607.
- [43] M. Meiler, R. Cordero-Soto, and S. K. Suslov, *Solution of the Cauchy problem for a time-dependent Schrödinger equation*, J. Math. Phys. 49 (2008) #7, 072102.
- [44] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical orthogonal polynomials of a discrete variable*, Springer-Verlag, Berlin, New York, 1991.
- [45] Y. Nogami and F. M. Toyama, *Nonlinear Schrödinger soliton in a time-dependent quadratic potential*, Phys. Rev. E 49, 5 (1994).
- [46] M. Ohta and G. Todorova, *Remarks on global existence and blow up for damped nonlinear Schrödinger equations*, Discrete Contin. Dyn. Syst. 23 (2009), no. 4, 1313-1325.
- [47] V. M. Perez-Garcia, P. J. Torres and V. K. Konotop, *Similarity transformations for nonlinear Schrödinger equations with time-dependent coefficients*, Physica D 221 (2006) 31-36.
- [48] S. A. Ponomarenko and G. P. Agrawal, *Do Solitonlike self-similar waves exist in nonlinear optical media?*, Phys. Rev. Lett. 97, 013901. (2006).
- [49] S. A. Ponomarenko and G. P. Agrawal, *Optical similaritons in nonlinear waveguides*, Optics letters 32, 12 (2007).
- [50] A.D. Polyanin, V.F. Zaisev, *Handbook of nonlinear partial differential equations*, Chapman Hall/CRC Pres, Boca Raton-London New York, (1912).
- [51] S. Raghavan and G. P. Agrawal, *Spatiotemporal solitons in inhomogeneous nonlinear media*, Opt. Commun. 180, 377, (2000).
- [52] V. N. Serkin and A. Hasegawa, *Novel soliton solutions of the nonlinear Schrödinger equation model*, Phys. Rev. Lett. 85, 4502 (2000).
- [53] V. Serkin, M. Matsumoto and T. Belyaeva, *Bright and dark solitary nonlinear Bloch waves in dispersion managed fiber systems and soliton lasers*, Opt. Comm. 196, 159 (2001).
- [54] E. Suazo, *Fundamental Solutions of some evolution equations*, Ph.D. Dissertation, Arizona State University. (September 2009).

- [55] E. Suazo and S. K. Suslov, *Soliton-like solutions for nonlinear Schrödinger equation with variable quadratic Hamiltonians*, Journal of Russian Laser Research. Vol. 33 (2010). 63-83.
- [56] E. Suazo and S. K. Suslov, *Cauchy problem for Schrödinger equation with variable quadratic Hamiltonians*, under preparation.
- [57] C. Sulem and P.L. Sulem, *The nonlinear Schrödinger equation*, Springer, New York, 1999.
- [58] S. K. Suslov, *On integrability of nonautonomous nonlinear Schrödinger equations*, A.M.S. Vol. 140 (2012). no. 9., 3067-3082.
- [59] V. I. Talanov, *Focusing of light in cubic media*. JETP Lett. 11 (1970), 199-201.
- [60] T. Tao, *Nonlinear dispersive equations: Local and global analysis*, CBMS Regional Conference Series in Mathematics, 2006
- [61] B. Tian, W. Shan, C. Zhang, G. Wei and Y. Gao, *Transformations for a generalized variable-coefficient nonlinear Schrödinger model from plasma physics, arterial mechanics and optical fibers with symbolic computation*, Eur. Phys. J. B 47 (2005), 329-332.
- [62] M. Tsutsumi, *Nonexistence of global solutions to the Cauchy problem for the damped nonlinear Schrödinger equations*. SIAM J. Math. Anal. 15 (1984), no. 2, 357-366.
- [63] Ming Wang, Wen-Rui Shan, Xing Lü, Yu-Shan Xue, Zhi-Qiang Lin and Bo Tian, *Soliton collision in a general coupled nonlinear Schrödinger system via symbolic computation*, Appl. Math. Comput. 219 (2013) 11258-11264.
- [64] M.I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. 87 (1983) 567-576.
- [65] F. Yu and Z. Yan, *New rogue waves and dark-bright soliton solutions for a coupled nonlinear Schrödinger equation with variable coefficients*, Appl. Math. Comput. 233 (2014) 351-358.
- [66] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Soviet. Phys. JETP. Vol. 34 (1972), no. 1.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PUERTO RICO, ARECIBO, P.O. BOX 4010, PUERTO RICO 00614-4010.

*E-mail address:* jose.escorcia@upr.edu

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF TEXAS RIO GRANDE VALLEY, 1201 W. UNIVERSITY DRIVE, EDINBURG, TEXAS, 78539-2999.

*E-mail address:* erwin.suazo@utrgv.edu